# Module-3: Relations and Functions

## Introduction

- Relations and Functions in real life give us the link between any two entities. In our daily life, we come across many patterns and links that characterize relation such as a relation between a father and a son, brother and sister, etc. In mathematics also, we come across many relations between numbers such as a number $x$ is less than $y$, line $L$ is parallel to line $m$, Bengaluru is the capital of Karnataka State, etc.
- Relation and function map elements of one set (domain) to the elements of another set (codomain)

## (*) Cartesian product of two sets

- ~~Explanation~~
- Definition :- Cartesian product is the set of all ordered pairs in which abscissa belongs to the first set and ordinate belongs to the second set

i.e., $\boxed{A \times B = \{(a, b) / a \in A, b \in B\}}$

$(x, y) \in A \times B \iff x \in A$ and $y \in B$ →Imp

NOTE:-

1. $A \times B \neq B \times A$, since $(a, b) \neq (b, a)$
2. $(a, b) = (c, d)$ iff $a = c$ and $b = d$
3. $A \times B = B \times A$ if $A = B$ or either $A$ or $B$ is $\phi$.
4. If $A = B$, $A \times A$ is well defined and is denoted by $A^2$.

   i.e., $A \times A = \{(a, b) / a \in A, b \in A\}$
5. For any non-empty set $A$, $A \times \phi = \phi = \phi \times A$.
6. For any set $S$, $|S|$ or $n(S)$ represents the number of elements in the set $S$, we have if

   $\boxed{|A| = n_1, |B| = n_2}$ then $\boxed{|A \times B| = n_1 \cdot n_2 = |A| \cdot |B|}$

## Solved problems

* Prove that for any three non-empty set

1. $A\times(B\cup C) = (A\times B)\cup(A\times C)$

$\cup$ – or

$\cap$ – and

**Proof:** Consider, LHS $= A\times(B\cup C)$

Let $(x,y)\in A\times(B\cup C)$

$\Leftrightarrow x\in A$ and $y\in(B\cup C)$

$\Leftrightarrow x\in A$ and $(y\in B$ or $y\in C)$

$\Leftrightarrow (x\in A$ and $y\in B)$ or $(x\in A$ and $y\in C)$

$\Leftrightarrow \{(x,y)\in(A\times B)\}$ or $\{(x,y)\in(A\times C)\}$

$\Leftrightarrow (x,y)\in\{(A\times B)\cup(A\times C)\}$

$\Rightarrow$ RHS $= (A\times B)\cup(A\times C)$

Thus, $A\times(B\cup C) = (A\times B)\cup(A\times C)$

2. $(A\cup B)\times C = (A\times C)\cup(B\times C)$

**Proof:** Let $(x,y)\in(A\cup B)\times C$

$\Leftrightarrow x\in(A\cup B)$ and $y\in C$

$\Leftrightarrow (x\in A$ or $x\in B)$ and $y\in C$

$\Leftrightarrow (x\in A$ and $y\in C)$ or $(x\in B$ and $y\in C)$

$\Leftrightarrow \{(x,y)\in(A\times C)\}$ or $\{(x,y)\in(B\times C)\}$

$\Leftrightarrow (x,y)\in(A\times C)\cup(B\times C)$

(3) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof: Let $(x, y) \in A \times (B \cap C)$

$\iff x \in A$ and $y \in (B \cap C)$

$\iff x \in A$ and $(y \in B$ and $y \in C)$

$\iff (x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\iff \{(x, y) \in (A \times B)\}$ and $\{(x, y) \in (A \times C)\}$

$\iff \{(x, y) \in (A \times B) \cap (A \times C)\}$

Thus, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(4) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

Proof: Let $(x, y) \in (A \cap B) \times C$

$\iff x \in (A \cap B)$ and $y \in C$

$\iff (x \in A$ and $x \in B)$ and $y \in C$

$\iff (x \in A$ and $y \in C)$ and $(x \in B$ and $y \in C)$

$\iff \{(x, y) \in (A \times B)\}$ and $\{(x, y) \in (B \times C)\}$

$\iff \{(x, y) \in (A \times B) \cap (B \times C)\}$

Thus, $(A \cap B) \times C = (A \times C) \cap (B \times C)$

# (*) Relations

A relation R from a non-empty set A to a non empty set B is a subset of the Cartesian product $A \times B$.

## → Image, Range, Domain and Codomain

$P = \{2, 4\}$ $Q = \{10, 20, 35\}$

$P \times Q = \{(p, q) : p \in P \text{ and } q \in Q\}$

$P \times Q = \{(2,10), (2,20), (2,35), (4,10), (4,20), (4,35)\}$

$R' = \{(p, q) : q \text{ is a multiple of } p,\ p \in P,\ q \in Q\}$

$R' = \{(2,10), (2,20), (4,20)\}$

Image → For every $(p, q) \in R'$ 'q' is a image of 'p'

→ 10 is image of 2
20 is image of 2
20 is image of 4

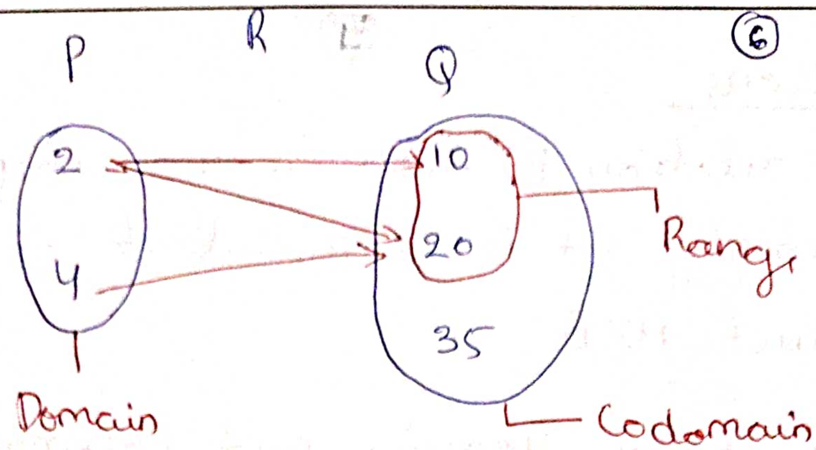
Range → q of (p, q) in R',
Range $= \{10, 20\}$

Domain → p of (p, q) in R'
Domain $= \{2, 4\}$

Range $\subseteq$ Q

Codomain → Whole Q
Codomain $= \{10, 20, 35\}$

ex:-




**NOTE:**

1. $P(S)$ denotes the power set of $S$. It contains the set of all subsets of $S$ has $2^n$ elements.
2. Let us suppose that $|A| = n_1, |B| = n_2 \Rightarrow |A \times B| = |A||B| = n_1 . n_2$
3. A relation $R \subseteq A \times B$, then there are $2^{n_1 n_2}$ relations from $A$ to $B$.
4. If $(a, b) \in R$ we say that $a$ is related to $b$ and write it as $aRb$.
5. A relation $R$ from a non empty set $A$ to itself is called a Binary Relation on $A$.

## (*) Types of Relations

Let us consider a non-empty set $A$.

(i) Reflexive relation $\rightarrow$ if $aRa$ i.e., $(a, a) \in R$, where $a \in A$.

(ii) Symmetric relation $\rightarrow$ if $aRb \Rightarrow bRa$ i.e., $(a, b) \in R \Rightarrow (b, a) \in R$ where $a, b \in A$

(iii) Transitive relation $\rightarrow$ if $aRb$ and $bRc \Rightarrow aRc$.
i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ where $a, b, c \in A$.

(iv) Anti - Symmetric $\rightarrow$ if $aRb$ and $bRa \Rightarrow a = b$.
i.e., $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$, where $a, b \in A$

(v) Equivalence $\rightarrow$ if relation is reflexive, symmetric and transitive

# Solved problem

(1) Let $A=\{2,4,6,8\}$ and $B=\{1,2,3\}$. Relations $R_1, R_2, R_3, R_4$ from A to B be defined as follows

(i) $aR_1b$ if $a \leq b$

(ii) $aR_2b$ if $a>b$

(iii) $aR_3b$ if a divides b

(iv) $aR_4b$ if b divides a

Sol:- Consider,

$$A \times B = \{(2,1)(2,2)(2,3)(4,1)(4,2)(4,3)(6,1)(6,2)(6,3)(8,1)(8,2)(8,3)\}$$

Therefore,

(i) $R_1 = \{(2,2)(2,3)\}$

(ii) $R_2 = \{(2,1)(4,1)(4,2)(4,3)(6,1)(6,2)(6,3)(8,1)(8,2)(8,3)\}$

(iii) $R_3 = \{(2,2)\}$

(iv) $R_4 = \{(2,1)(2,2)(4,1)(4,2)(6,1)(6,2)(6,3)(8,1)(8,2)\}$

2) Let A and B be finite sets with $|B| = 3$. If there are 4096 relations from A to B, what is $|A|$?

Sol:- We know that, if $|A| = m$ and $|B| = n$, then there are $2^{mn}$ relations from A to B.

- Given, $n = 3$ and $2^{mn} = 4096$

Thus, $2^{3m} = 4096$ (Take $\log_e$ on both sides)

or $\log_e 2^{3m} = \log_e 4096$

or $3m \log_e 2 = \log_e 4096$ ($\log a^m = m \log a$)

or $m = \dfrac{\log_e 4096}{3 \log_e 2} = 4.$

Thus $|A| = 4$

3. Show that the following relation R defined on the set of all integers Z is an equivalence relation.
$R = \{(x,y) / x, y \in Z$ and $(x-y)$ is an even integer$\}$

Sol:- To show that R is an equivalence relation, we shall show that R is reflexive, symmetric and transitive.

Given, $xRy$ if $(x-y)$ is an even integer.

(i) $xRx \Rightarrow x-x=0$ is even. Hence R is reflexive.

(ii) $xRy \Rightarrow (x-y)$ is an even integer

$\therefore yRx \Rightarrow (y-x) \Rightarrow -(x-y)$ is also an even integer

i.e., $xRy \Rightarrow yRx$. Hence R is symmetric

(iii) $xRy \Rightarrow (x-y)$ is even. i.e., $x-y = 2m$ (say), $m \in Z$

$yRz \Rightarrow (y-z)$ is even. i.e., $y-z = 2n$ (say), $n \in Z$

Now, $x-z = (x-y)+(y-z) = 2m+2n = 2(m+n)$ is even

$\therefore (x-z)$ is even $\Rightarrow xRz$

i.e., $xRy$ and $yRz \Rightarrow xRz$. Hence R is transitive

Therefore, R is an equivalence relation.

## (*) properties of Relations

The properties of Relations ~~is~~ concerns on the aspect of matrix representation and graphical representation of relations. Relations satisfying certain properties is highly significant in several computer applications. Matrix and graphical representation of relation is highly useful in visualizing several concepts of applications.

## (*) Matrix representation of a relation - Zero - One Matrices

- If $A=\{a_1, a_2, --- a_m\}$ and $B=\{b_1, b_2, -- b_n\}$ are two finite sets, then the cartesian product of A and B denoted by $A \times B$ is given by

$$A \times B = \{(a_i, b_j) / 1 \le i \le m, 1 \le j \le n\}$$

- This is represented in the form of a $m \times n$ matrix as follows

$$M(A \times B) = \begin{bmatrix} (a_1, b_1) & (a_1, b_2) & (a_3, b_3) & --- & (a_1, b_n) \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & --- & (a_2, b_n) \\ --- & --- & --- & --- & --- \\ --- & --- & --- & --- & --- \\ (a_m, b_1) & (a_m, b_2) & (a_m, b_3) & --- & (a_m, b_n) \end{bmatrix}$$

Imp

Matrix representation of a relation R denoted by M(R) is defined as follows.

$$M(R) = [m_{ij}] \text{ where } m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

- Here $m_{ij}$ represents the element in the $i^{th}$ row and $j^{th}$ column of the matrix. It is evident from the definition that M(R) contains 0 and 1 only as its elements.
- Hence, M(R) is called a Zero-one matrix.

NOTE- (1) if $R = A \times B$ itself, then M(R) is a matrix with every element being 1 and is called a one all matrix

(2) if $R = \phi$, a null set then M(R) is a null matrix with every element being 0.

## Examples

1. Let $A=\{a_1, a_2\}$ ; $B=\{b_1, b_2\}$. Then we have, $A\times B=\{(a_1,b_1)(a_1,b_2)(a_2,b_1)(a_2,b_2)\}$. Also let us consider the following relations $R_1=\{(a_1,b_1)(a_2,b_2)\}$, $R_2=\{(a_1,b_2)(a_2,b_1)\}$ $R_3=\{(a_1,b_1)(a_1,b_2)(a_2,b_2)\}$. Write these relations in matrix form.

S:- We have the following matrix representation

$$M(A\times B)=\begin{bmatrix}(a_1,b_1) & (a_1,b_2)\\ (a_2,b_1) & (a_2,b_2)\end{bmatrix}$$

$$M(R_1)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$M(R_2)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$M(R_3)=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

2. Let $A=\{a, b, c\}$ and $B=\{1,2\}$

$R_1=\{(a,1)(b,2)(c,1)\}$, $R_2=\{(a,2)(b,2)(c,2)\}$,

$R_3=\{(a,1)(b,1)(c,1)(c,2)\}$

Sol: $A\times B=\{(a,1)(a,2)(b,1)(b,2)(c,1)(c,2)\}$

$$M(A\times B)=\begin{bmatrix}(a,1) & (a,2)\\(b,1) & (b,2)\\(c,1) & (c,2)\end{bmatrix}$$

$$\therefore M(R_1)=\begin{bmatrix}1 & 0\\0 & 1\\1 & 0\end{bmatrix}$$

$$M(R_2)=\begin{bmatrix}0 & 1\\0 & 1\\0 & 1\end{bmatrix}$$

$$M(R_3)=\begin{bmatrix}1 & 0\\1 & 0\\1 & 1\end{bmatrix}$$

(15)

(3) Let $M(R) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

write down the associated relation R.

S:- • $M(R) = 3 \times 4$ matrix

• Let us ~~know~~ take $A = \{a_1, a_2, a_3\}$ & $B = \{b_1, b_2, b_3, b_4\}$.

• The associated $R = \{(a_1, a_1)(a_1, b_3)(a_2, b_2)(a_2, b_4)$
$(a_3, b_1)(a_3, b_2)\}$

## (*) Graphical representation of a relation

→ **Directed Graph or Digraph**

- A relation $R$ on a finite set $A$ $(R \subseteq A \times A)$ can be represented pictorially as follows :
- Step : Let $a_1, a_2$ be the elements of the set $A$

(1)


- Here, $a_1$ and $a_2$ are written within the circle. The circle is called vertex or node.

- Here, $a_1, a_2 \in A$ and $(a_1, a_2) \in R$ are connected by an arc (or straight line) and an arrow in the direction from $a_1$ to $a_2$ is also indicated. This arrow is called an 'edge'.

- Digraph $G = (V, E)$, where $V$ = set of all vertices
  $E$ = Edges (Arrows)

- Indegree ($I_d$): The number of arrows terminating at a vertex

- outdegree ($O_d$): The number of arrows leaving a vertex

## Solved problems

1. Let $A = \{1, 2, 3, 4, 6\}$ and $R$ be a relation on $A$ defined by $aRb$ if and only if 'a' is a multiple of b. Represent $R$ as a set of ordered pairs. Draw the diagraph and matrix representation of R.

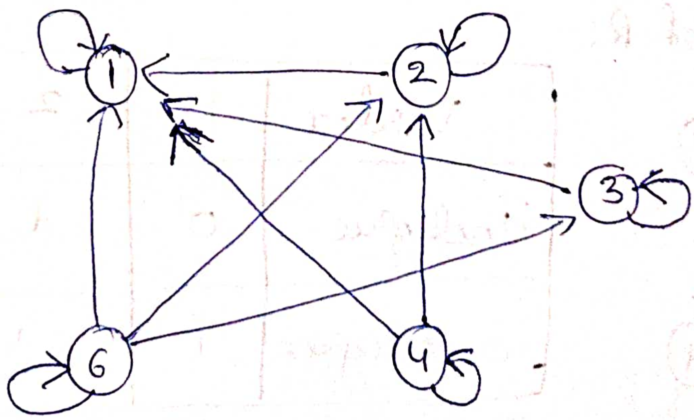
Sol: Given, $A = \{1, 2, 3, 4, 6\}$

- $aRb$ iff 'a' is a multiple of b

$\therefore R = \{(1,1)(2,1)(2,2)(3,1)(3,3)(4,1)(4,2)(4,4)(6,1)(6,2)$
$(6,3)(6,6)\}$

- $$M(R) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

- Digraph of R is as follows



2 Let $A=\{1,2,3,4\}$, let R be a relation on A defined by $xRy$ iff $x/y$ and $y=2x$. Write down the following.

(i) R as a relation of set of ordered pairs

(ii) Digraph of R

(iii) Indegrees and outdegrees of the vertices in the digraph.

Sol: • $A\times A=\{(1,1)(1,2)(1,3)(1,4)(2,1)(2,2)(2,3)(2,4)(3,1)(3,2)(3,3)(3,4)(4,1)(4,2)(4,3)(4,4)\}$

• Let, $x/y \Rightarrow y=Kx$, where K is an integer

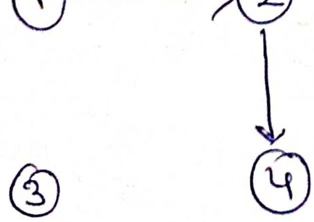
• The set of ordered pair satisfying this condition are

$\{(1,1)(1,2)(1,3)(1,4)(2,2)(2,4)(3,3)(4,4)\}$

• Further $xRy$ iff $x/y$ and $y=2x$ is being satisfied only by two ordered pairs $(1,2)(2,4)$

Thus $R=\{(1,2)(2,4)\}$

Digraph of R:




| Vertex | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Indegree | 0 | 1 | 0 | 1 |
| outdegree | 1 | 1 | 0 | 0 |

## (*) Composition of relations

- Let A, B, C be three non empty sets. Further let $R_1$ be a relation from A to B and $R_2$ be a relation from B to C. Then

$$R_1 \circ R_2 = \{(a,c) / a \in A,\ c \in C \text{ and } \exists\, b \in B \text{ such that } (a,b) \in R_1 \text{ and } (b,c) \in R_2\}$$

Note:- ① If R is a relation on a non empty set A,

$R \circ R = R^2$, $R^2 \circ R = R^3$ and so on.

② $M(R_1) \cdot M(R_2) = M(R_1 \circ R_2)$ and in particular

$$[M(R)]^2 = M(R^2)$$

③ If $A = \{1,2,3,4\}$ and R, S are relations on A defined by $R = \{(1,2)(1,3)(2,4)(4,4)\}$

$S = \{(1,1)(1,2)(1,3)(1,4)(2,3)(2,4)\}$

Find $R \circ S$, $S \circ R$, $R^2$ and $S^2$.

Sol:- ⓘ $(1,2) \in R$, $(2,3)$ and $(2,4) \in S \Rightarrow (1,3)$ and $(1,4) \in R \circ S$

$$R \circ S = \{(1,3)(1,4)\}$$

(ii) $(1,1) \in S$, $(1,2)$ and $(1,3) \in R \Rightarrow (1,2)$ and $(1,3) \in S \circ R$

$(1,2) \in S$, $(2,4) \in R \Rightarrow (1,4) \in S \circ R$

$(1,4) \in S$, $(4,4) \in R \Rightarrow (1,4) \in S \circ R$

$(2,4) \in S$, $(4,4) \in R \Rightarrow (2,4) \in S \circ R$

$\therefore S \circ R = \{(1,2)(1,3)(1,4)(2,4)\}$

(iii) $R^2 = R \circ R$

$(1,2) \in R$, $(2,4) \in R \Rightarrow (1,4) \in R^2$

$(2,4) \in R$, $(4,4) \in R \Rightarrow (2,4) \in R^2$

$R^2 = \{(1,4)(2,4)\}$

(iv) $S^2 = S \circ S$

$(1,1) \in S$, $(1,1) \in S \Rightarrow (1,1) \in S^2$

$(1,1) \in S$, $(1,2), (1,3) \& (1,4) \in S \Rightarrow (1,2)(1,3)(1,4) \in S^2$

$(1,2) \in S$, $(2,3)$ and $(2,4) \in S \Rightarrow (1,3)(1,4) \in S^2$

$\therefore S^2 = \{(1,1)(1,2)(1,3)(1,4)\}$

4. Let $R = \{(1,2)(1,3)(2,4)(3,2)\}$ be a relation on $A = \{1,2,3,4\}$.

(i) Write down the relation matrix $M(R)$ of $R$

(ii) Compute $[M(R)]^2$. Hence obtain $R^2$.

Sol: Given, $R = \{(1,2)(1,3)(2,4)(3,2)\}$

$$\therefore M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$[M(R)]^2 = M(R).M(R)$. (Matrix multiplication)

$$[M(R)]^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By using the property

$$[M(R)]^2 = M(R^2)$$

So that

$$M(R^2) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore R^2 = \{(1,2)(1,4)(3,4)\}$

5) Let $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$ and $C = \{p, q, r, s\}$.
Consider $R_1 = \{(1,x)(2,w)(3,z)\}$ a relation from A to B.
$R_2 = \{(w, p)(z,q)(y,s)(x,p)\}$ a relation from B to C.

(i) What is the composite relation $R_1 o R_2$ from A to C?

(ii) Write relation matrices $M(R_1)$, $M(R_2)$ and $M(R_1 o R_2)$

(iii) verify that $M(R_1) \cdot M(R_2) = M(R_1 o R_2)$

Sol:- We have, $R_1 = \{(1,x)(2,w)(3,z)\}$

$R_2 = \{(w,p)(z,q)(y,s)(x,p)\}$

(i) $R_1 \circ R_2 = \{(1,p)(2,p)(3,q)\}$

(ii)
$$M(R_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M(R_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M(R_1 \circ R_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(iii) Consider.

$$M(R_1) \cdot M(R_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= M(R_1 \circ R_2)$$

Thus, $M(R_1) \cdot M(R_2) = M(R_1 \circ R_2)$ is verified

## (*) Partial orders

- A relation R on set A is said to be a partial ordering relation or a partial order on A if
  (i) R is reflexive (ii) R is antisymmetric, and
  (iii) R is transitive, on A.

- A set A with a partial order R defined on it is called a partially ordered set or an ordered set or a poset, and is denoted by the pair (A, R)

- Partially ordered relation R on A is called a totally ordered relation on A, if, for all $x, y \in A$, either $xRy$ or $yRx$. In such cases the poset (A, R) is called a totally ordered set.

Ex:- $(\mathbb{Z}, \leq)$ } → poset
$(\mathbb{Z}, \geq)$

$(\mathbb{R}, \leq)$ } → totally ordered set.
$(\mathbb{R}, \geq)$

Note:- Every total order is a partial order, but not every partial order is a total order.

## (*) Hasse diagram for a partially ordering relation

- The partially ordering relation is reflexive, anti-symmetric and transitive the graph of this relation can be represented in a simpler way by following certain conventions. Such a pictorial representation of a partial order is called the Hasse diagram.

We follow the following conventions.

(1) Since R is reflexive on A, $\forall x \in A$, $xRx$. Digraph of R will have a loop at each vertex. In Hasse diagram these loops are not exhibited as a convention.

(2) Since R is also transitive, $xRy$ and $yRx \Rightarrow xRz$. In the digraph of R edges from x to y, y to z and also from x to z are being shown. In Hasse diagram from x to z is not indicated as a the edge from x to z is not indicated as a convention.
(i.e., Connection to each of the connecting pair is left as a convention)

(3) The vertex are represented by dots(·) and if $xRy$, the edge from x to y is drawn upwards as a straight line and arrows is not indicated as a convention. Hasse diagram is also called a poset diagram.

**Note:-** Hasse diagram can be drawn for any relation which is antisymmetric and transitive, not necessarily reflexive. Hence we can say that two unequal relations may have the same Hasse diagram.

**Ex:-**

(1) $A=\{1,2,3\}$ and $R=\{(1,1)(2,2)(3,3)(1,2)(1,3)(2,3)\}$

Sol:- • R is reflexive since $(x,x)\in R \ \forall x \in A$.

• R is antisymmetric: when $(x,y)\in R$ and $x\neq y$, $(y,x)\notin R$
when $(x,y)\in R$ and $(y,x)\in R \Rightarrow x=y$

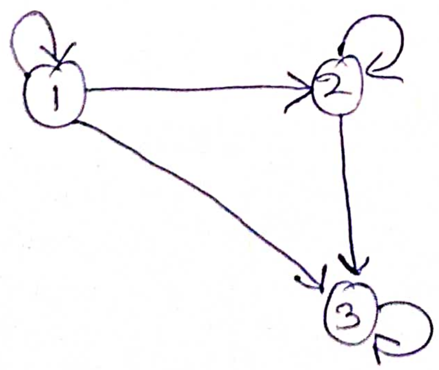
• R is transitive for the pairs.

$(1,1)(1,2)$; $(1,1)(1,3)$; $(2,2)(2,3)$; $(1,3)(3,3)$; $(2,3)(3,3)$

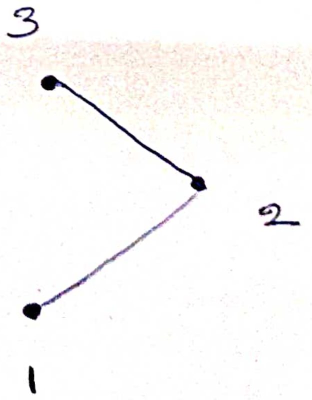
Further $(1,2)(2,3)\in R$ and $(1,3)\in R$

$\therefore$ R is a partially ordered relation

Digraph of R




Hasse diagram of R



2. $A=\{1,2,3,4\}$ , $R=\{(1,1)(1,2)(1,3)(1,4)(2,2)(2,4)(3,3)(4,4)\}$

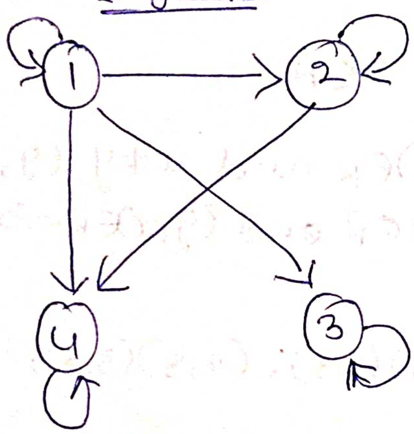
Sol: • R is reflexive since $(1,1),(2,2),(3,3)$ and $(4,4) \in R$

• R is anti-symmetric since $(x,y) \in R$, $(y,x) \in R \Rightarrow x=y$

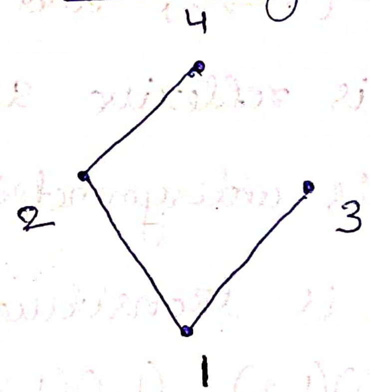
• R is transitive since $(1,2)(2,4) \in R$ and $(1,4) \in R$

$\therefore$ R is partially ordered relation

Digraph




Hasse diagram

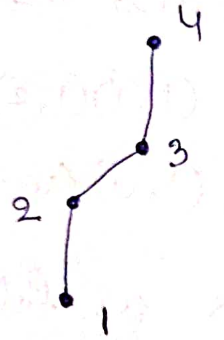


# Worked problems

① Given $A = \{1,2,3,4\}$ and the relation $R$ on $A$ is defined by $xRy$ if $x \le y$. Draw the Hasse diagram of $(A,R)$.

**S:-** We have the relation $R$ as follows.

$R = \{(1,1)(1,2)(1,3)(1,4)(2,2)(2,3)(2,4)(3,3)(3,4).(4,4)\}$

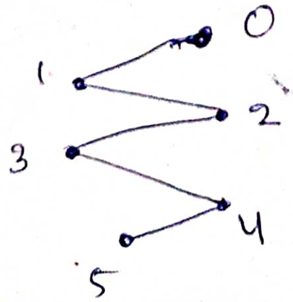



② Draw the Hasse diagram for '$\ge$' relation defined on the set $\{0,1,2,3,4,5\}$ and we have $xRy$ iff $x \ge y$

**S:-** Let $A = \{0,1,2,3,4,5\}$

$\therefore$ 5R4, 4R3, 3R2, 2R1, 1R0

$\therefore$



(3) If R is a relation on $A = \{1,2,3,4\}$ defined by $xRy$ if $x$ divides $y$. Prove that $(A,R)$ is a Poset. Draw its Hass diagram.

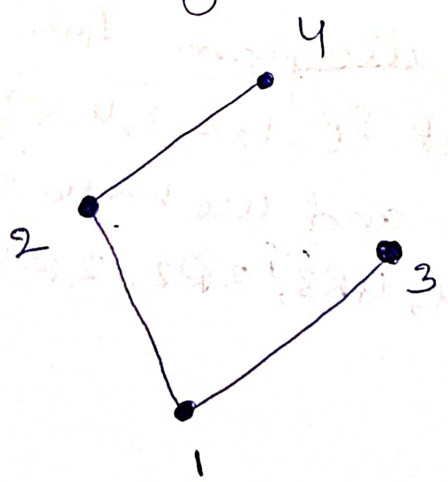
S:- Given, $A = \{1,2,3,4\}$

$R = \{(x,y) / x|y\}$ or $R = \{(x,y) / x \text{ divides } y\}$

$\therefore R = \{(1,1)(1,2)(1,3)(1,4)(2,2)(2,4)(3,3)(4,4)\}$

- R is reflexive, since $(x,x) \in R \ \forall x \in A$
- R is anti-symmetric, since $(x,y) \in R$ and $x \neq y$, $(y,x) \notin R$
- R is transitive for the pairs: $(1,1)(1,2)$; $(1,1)(1,3)$; $(1,1)(1,4)$; $(1,2)(2,2)$; $(1,3)(3,3)$; $(1,4)(4,4)$; $(1,2)(2,4)$

$\therefore$ The Hasse diagram of R is as follows



(4) Draw the Hasse diagram representing the positive divisors of 36.

Sol:- • Let A be the set consisting of all divisors of 36.

• i.e., $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

• The relation R of divisibility (i.e., aRb iff a|b) is a partial order on this set. The Hasse diagram for this partial order is required here.

• We note that, under R.

1 is related to all elements of A,

2 is related to 2, 4, 6, 12, 18, 36;

3 is related to 3, 6, 9, 12, 18, 36;

4 is related to 4, 12, 36;

6 is related to 6, 12, 18, 36;

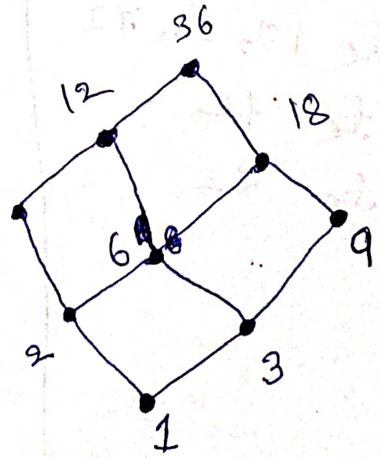
9 is related to 9, 18, 36;

12 is related to 12 and 36;

18 is related to 18 and 36;

36 is related to 36.

• The Hasse diagram is as follows



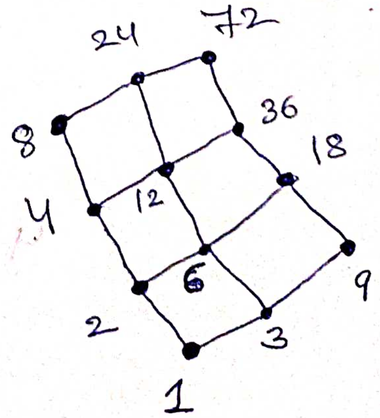
(5) Draw the Hasse diagram for all positive integer divisors of 72.

Sol: Let A be the set consisting of all divisors of 72.

$A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72\}$

- The relation R of divisibility (i.e., aRb iff a|b) is a partial order on this set. The Hasse diagram for this partial order is required here.
- We note that under R
- 1 is related to all elements of A.
- 2 is related to 2, 4, 6, 8, 12, 18, 24, 36, 72
- 3 is related to 3, 6, 9, 12, 18, 24, 36, 72
- 4 is related to 4, 8, 12, 24, 36, 72
- 6 is related to 6, 12, 18, 24, 36, 72
- 8 is related to 8, 24, 72
- 9 is related to 9, 18, 36, 72
- 12 is related to 12, 24, 36, 72
- 18 is related to 18, 36, 72
- 24 is related to 72
- 36 is related to 72

∴ The Hasse diagram



# (*) Partitions and Equivalence class

## Definition of partition of a set

- Let $A$ be a non empty set and let the set $P=\{A_1, A_2, A_3, ----A_n\}$ where $A_i$'s ($i=1$ to $n$) are subset of $A$. The set $P$ is called a partition of $A$ if the following conditions are satisfied.

(i) $A_1 \cup A_2 \cup A_3 \cup --- \cup A_n = A$

(ii) $A_i \cap A_j = \phi$ where $i \neq j$

i.e., the union of the subsets of $A$ should be $A$ and the subsets are mutually disjoint.

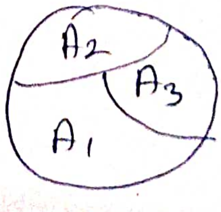
### Examples

(1) Let $A=\{a, b, c, d, e, f\}$ and the subsets of $A$ be,
$A_1=\{a, c, d\}$, $A_2=\{b, e\}$, $A_3=\{f\}$

Sol:- We note the following:
$A_1 \cup A_2 \cup A_3 = \{a, b, c, d, e, f\} = A$ and
$A_1 \cap A_2 = \phi$, $A_2 \cap A_3 = \phi$, $A_3 \cap A_1 = \phi$

$\therefore P=\{A_1, A_2, A_3\}$ is a partition of $A$.

i.e.,



## Definition of Equivalence class

Let $A$ be a non-empty set and $R$ be an equivalence relation on $A$. Equivalence class of an element $x \in A$ is denoted by $[x]$ and is defined as follows by the rule method,

$$[x] = \{ y \in A / yRx \text{ or } (y, x) \in R \}$$

### Example

① Let $A = \{1, 2, 3, 4\}$ and the equivalence relation $R$ on $A$ be, $R = \{(1,1)(2,2)(3,3)(4,4)(1,2)(2,1)\}$

Sol:- $[1] = \{1, 2\}$, since $(1,1)(2,1) \in R$

$[2] = \{1, 2\}$, since $(1,2)(2,2) \in R$

$[3] = \{3\}$, since $(3,3) \in R$

$[4] = \{4\}$, since $(4,4) \in R$

Property:- An equivalence relation $R$ on a non-empty set $A$ induces a partition of $A$ and vice-versa.

## Solved problems

1. If $R = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$ is defined on the set $A = \{1,2,3,4\}$ determine the partition induced.

Sol:-

$[1] = \{1,2\}$ since $(1,1)(2,1) \in R$

$[2] = \{1,2\}$ since $(1,2)(2,2) \in R$

$[3] = \{3,4\}$ since $(3,3)(3,4) \in R$

$[4] = \{3,4\}$ since $(4,4)(4,3) \in R$

$\therefore$ Required partition $P = \{\{1,2\}\{3,4\}\}$

2. Let $A = \{a,b,c,d,e,f,g\}$ and consider the partition $P = \{\{a,c,d\}, \{b\}, \{e,g\}, \{f\}\}$. Determine the corresponding equivalence relation

Sol:- Consider;

(i) $\{a,c,d\} \Rightarrow (a,a)(c,c)(d,d)(a,c)(a,d)(c,a)(c,d)(d,a)(d,c) \in R$

(ii) $\{b\} \Rightarrow (b,b) \in R$

(iii) $\{e,g\} \Rightarrow (e,e)(g,g)(e,g)(g,e) \in R$

(iv) $\{f\} \Rightarrow (f,f) \in R$

$\therefore R = \{(a,a)(b,b)(c,c)(d,d)(e,e)(g,g)(f,f)(a,c)(c,a)(c,d)(d,c)(a,d)(d,a)(e,g)(g,e)\}$

(3) Let $A = \{1, 2, 3, 4, 5\}$. Define a relation $R$ on $A \times A$ by $(x_1, y_1) R (x_2, y_2)$ if and only if $x_1 + y_1 = x_2 + y_2$.

(i) verify that $R$ is an equivalence relation on $A \times A$.

(ii) Determine the equivalence classes $[(1,3)]$, $[(2,4)]$ and $[(1,1)]$

(iii) Determine the partition of $A \times A$ induced by $R$.

Sol:- (i) • For all $(x, y) \in A \times A$, we have $x + y = x + y$

i.e., $(x, y) R (x, y) \Rightarrow R$ is reflexive

• For any $(x_1, y_1), (x_2, y_2) \in A \times A$, let $(x_1, y_1) R (x_2, y_2)$ then $(x_1 + y_1) = (x_2 + y_2)$. This gives $x_2 + y_2 = x_1 + y_1$ which means $(x_2, y_2) R (x_1, y_1)$. Therefore, $R$ is Symmetric

• For any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in A \times A$, let $(x_1, y_1) R (x_2, y_2)$ and $(x_2, y_2) R (x_3, y_3)$. Then $x_1 + y_1 = x_2 + y_2$ and $x_2 + y_2 = x_3 + y_3$. This gives $x_1 + y_1 = x_3 + y_3$, which means $(x_1, y_1) R (x_3, y_3)$. Therefore, $R$ is transitive

Thus, $R$ is reflexive, symmetric and transitive.

Therefore, $R$ is an equivalence relation.

(ii) We note that

$[(1,3)] = \{(x,y) \in A\times A \mid (x,y) R (1,3)\}$

$= \{(x,y) \in A\times A \mid x+y = 1+3 \text{ or } x+y = 4\}$

- $[(1,3)] = \{(1,3)(2,2)(3,1)\}$ ($\because A = \{1,2,3,4,5\}$)
- $[(2,4)] = \{(1,5)(2,4)(3,3)(4,2)(5,1)\}$
- $[(1,1)] = \{(1,1)\}$.

(iii) To determine the partition induced by R, we have to find the equivalence classes of all elements $(x,y)$ of $A\times A$ w.r.t R.

i.e, $A\times A = \{(1,1)(1,2)(1,3)(1,4)(1,5)(2,1)(2,2)(2,3)(2,4)(2,5)(3,1)(3,2)(3,3)(3,4)(3,5)(4,1)(4,2)(4,3)(4,4)(4,5)(5,1)(5,2)(5,3)(5,4)(5,5)\}$

- $[(1,1)] = \{(1,1)\}$
- $[(1,2)] = \{(1,2)(2,1)\} = [(2,1)]$
- $[(1,3)] = \{(1,3)(2,2)(3,1)\} = [(2,2)]$ Also $[(3,1)]$
- $[(1,4)] = \{(1,4)(2,3)(3,2)(4,1)\} = [(4,1)]$ Also $[(2,3)] = [(3,2)]$
- $[(1,5)] = \{(1,5)(5,1)(3,3)(4,2)(2,4)\} = [(5,1)]$ Also $[(3,3)]$

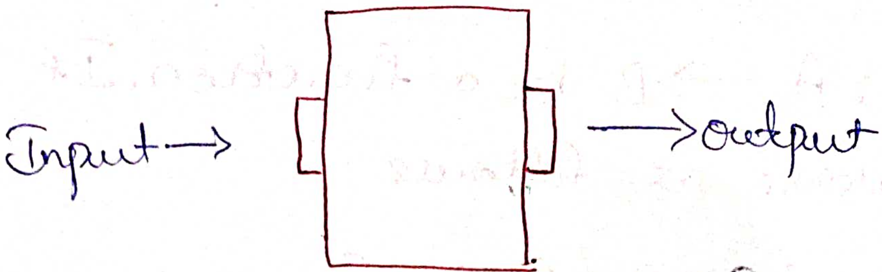
Also $[(4,2)] = [(2,4)]$

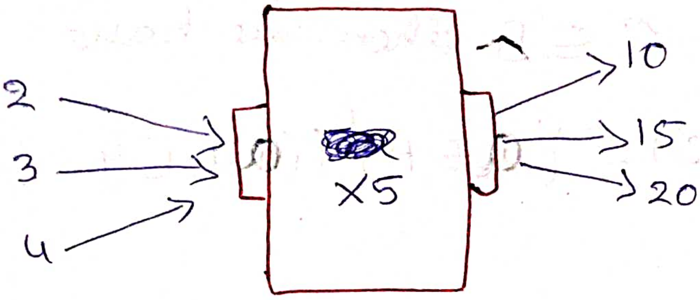
- $[(2,5)] = \{(5,2)(2,5)(3,4)(4,3)\} = [(5,2)]$ Also $[(3,4)] = [(4,3)]$
- $[(3,5)] = \{(3,5)(5,3)(4,4)\} = [(5,3)]$ Also $[(4,4)]$
- $[(4,5)] = \{(4,5)(5,4)\} = [(5,4)]$
- $[(5,5)] = \{(5,5)\} = [(5,5)]$

$\therefore$ the partition of $A \times A$ induced by $R$ is represented by

$$A \times A = \{[(1,1)] \cup [(1,2)] \cup [(1,3)] \cup [(1,4)] \cup [(1,5)] \cup [(2,5)] \cup [(3,5)] \cup [(4,5)] \cup [(5,5)]\}$$

# (*) Functions




Function

$f(x) = 5x$, where $x = 2, 3, 4$

## Definition

A function f from a set A to a set B is a specific type of relation for which every element a of set A has one and only one image b in set B.

i.e., $f: A \longrightarrow B$, where $f(a) = b$, $(a, b) \in f$

~~Note: Inverse function~~

~~Let $f: A \longrightarrow B$ be a function. If $b \in B$, then~~

~~$f^{-1}(b) = \{x \in A / f(x) = b\}$~~

**NOTE:-**

- Let $f: A \longrightarrow B$ be a function. If $b \in B$, we define the inverse as follows

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}$$

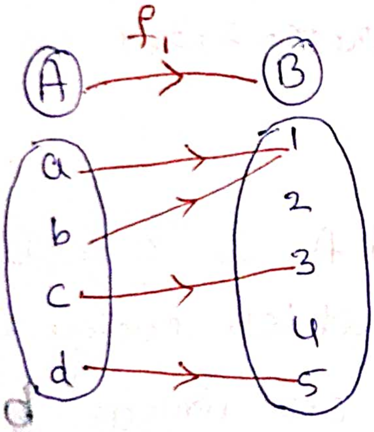
Further, if $C \subseteq B$ then we have

$$f^{-1}(C) = \{a \in A \mid f(a) \in C\}$$

**Examples**

Let $A = \{a, b, c, d\}$ ; $B = \{1, 2, 3, 4, 5\}$

(i)

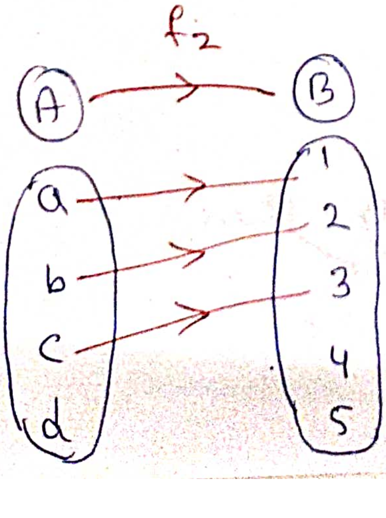


- $f_1 : \{(a,1)(b,1)(c,3)(d,5)\}$

  or

  $f_1(a) = 1$ , $f_1(b) = 1$, $f_1(c) = 3$, $f_1(d) = 5$
- $f_1$ is a function since $a, b, c, d$ appears only once in the ordered pair
- Range of $f_1 = \{1, 3, 5\}$
- $f_1^{-1}(1) = \{a, b\}$ , $f_1^{-1}(3) = \{c\}$, $f_1^{-1}(5) = \{d\}$

(ii)



- $f_2$ is not a function since the element d of the set A is left over

## (*) Types of functions :

### ① one-one function (or) Injective

A function $f: A \rightarrow B$ is defined to be one-one (or injective), if distinct elements of A are mapped into distinct elements of B.

i.e., $\forall\, a_1, a_2 \in A,\ a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

(or)

$\forall\, a_1, a_2 \in A,\ a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$

### ② onto function (or) Surjective

A function $f: A \rightarrow B$ is said to be onto function (or surjective), if every elements of B is the image of some element of A.

i.e.,

$\forall\, b \in B, \exists\, a \in A \ni f(a) = b$

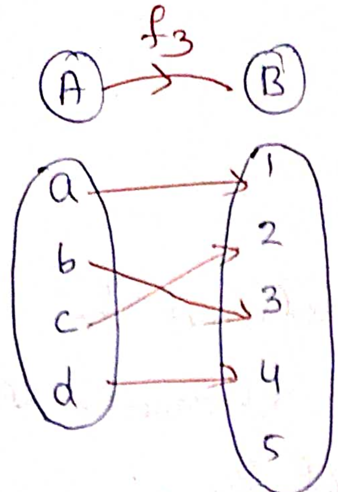
**Note:-** If A and B are finite sets with $|A| = m$ and $|B| = n$ where $m \geq n$ then

① The number of one-to-one functions is $P(n, m) = \dfrac{n!}{(n-m)!}$

② The no of onto functions is $P(n,m) = \sum_{K=0}^{n} (-1)^K \left({}^nC_{n-K}\right)(n-K)^m$

(44)

(iii)

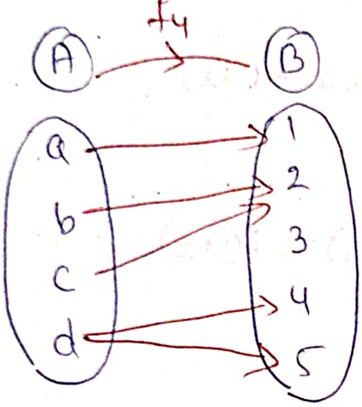



$f_3 = \{(a,1)(b,3)(c,2)(d,4)\}$

$f_3$ is a function

Range of $f_3 = \{1,2,3,4\}$

(iv)




$f_4$ is not a function since the element d of the set A has two images, i.e., $f_4(d)=4$ and $f_4(d)=5$.

NOTE:-

Number of functions from a set A to B

Let $|A| = m$ and $|B| = n$

$A = \{a_1, a_2, \ldots a_m\}$, $B = \{b_1, b_2, \ldots b_n\}$

A function $f_1$ from A to B is of the form,

$f_1 = \{(a_1, b)(a_2, b), \ldots (a_n, b)\}$

where b can be any of the n elements $b_1, b_2, \ldots b_n$ of B.

i.e., $n \times n \times n \ldots$ m times $= n^m = (|B|)^{|A|}$

$\boxed{\text{The no of functions from A to B is } (|B|)^{|A|}}$

# Worked problems

① Let $f: R \to R$ be defined by $f(x) = \begin{cases} 3x-5 & \text{if } x>0 \\ -3x+1 & \text{if } x \le 0 \end{cases}$

(i) Determine $f(0)$, $f(1)$, $f(-1)$, $f(5/3)$ and $f(-5/3)$

(ii) Find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$, $f^{-1}(-6)$.

(iii) What are $f^{-1}([-5,5])$ and $f^{-1}([-6,5])$

Sol:- (i) Given, $f(x) = 3x-5$ if $x>0$

$\therefore f(1) = 3(1)-5 = -2$

$f\left(\frac{5}{3}\right) = 3\left(\frac{5}{3}\right)-5 = 0$

Also, $f(x) = -3x+1$ if $x \le 0$

$f(0) = -3(0)+1 = 1$

$f(-1) = -3(-1)+1 = 4$

$f\left(\frac{-5}{3}\right) = -3\left(-\frac{5}{3}\right)+1 = 6$

(ii) From the definition of the given $f$, we find that $f(x)=0$ only when $x=\frac{5}{3}$. (observe that $f(x) \neq 0$ for $x \le 0$).

$\therefore \quad f^{-1}\{0\} = x$

$\Rightarrow f^{-1}(0) = \frac{5}{3}$

- $f^{-1}(1) = \{x \in R \mid f(x) = 1\}$

  i.e., when $x > 0$, $3x - 5 = 1$ or $3x = 6 \Rightarrow x = 2$

  when $x \leq 0$, $-3x + 1 = 1$ or $-3x = 0 \Rightarrow x = 0$

  $\therefore f^{-1}(1) = \{0, 2\}$

- $f^{-1}(-1) = \{x \in R / f(x) = -1\}$

  ie, when $x > 0$, $3x - 5 = -1$ or $3x = +4 \Rightarrow x = \frac{4}{3}$

  ~~when $x \leq 0$, $-3x + 1 = -1$ or $-3x = -2 \Rightarrow x = \frac{2}{3}$~~

  $\therefore f^{-1}(-1) = \{4/3\}$ ($\because f(x) \neq -1$ when $x \leq 0$)

- $f^{-1}(3) = \{x \in R / f(x) = 3\}$

  ie, when $x > 0$, $3x - 5 = 3$ or $3x = 8 \Rightarrow x = 8/3$

  when $x \leq 0$, $-3x + 1 = 3$ or $-3x = 2 \Rightarrow x = \frac{-2}{3}$

  $\therefore f^{-1}(3) = \{\frac{-2}{3}, \frac{8}{3}\}$

- $f^{-1}(-3) = \{x \in R \mid f(x) = -3\}$

  when $x > 0$, $3x - 5 = -3$ or $3x = 2 \Rightarrow x = 2/3$

  $\therefore f^{-1}(-3) = \{2/3\}$ ($\because f(x) \neq -3$ when $x \leq 0$)

- $f^{-1}(-6) = \phi$, because $f(x) \neq -6$ for any $x \in R$.

(iii) $f^{-1}([-5,5]) = \{x \in R \,/\, f(x) \in [-5,5]\}$

$= \{x \in R \,/\, -5 \leq f(x) \leq 5\}$

• When $x > 0$, we have $f(x) = 3x - 5$

$\therefore\ -5 \leq 3x - 5 \leq 5$

$\Rightarrow -5+5 \leq 3x \leq 5+5$

$\Rightarrow\ 0 \leq 3x \leq 10$

$\Rightarrow\ 0 \leq x \leq \frac{10}{3}$

• When $x \leq 0$, we have $f(x) = -3x + 1$

$\therefore\ -5 \leq f(x) \leq 5$

$\Rightarrow -5 \leq -3x+1 \leq 5$

$\Rightarrow -6 \leq -3x \leq 4$

$\Rightarrow 2 \geq x \geq -\frac{4}{3}$

or $-\frac{4}{3} \leq x \leq 2$

$\therefore\ f^{-1}([-5,5]) = \{x \in R \,/\, -4/3 \leq x \leq 2 \text{ or } 0 \leq x \leq 10/3\}$

$= \{x \in R \,/\, -4/3 \leq x \leq 10/3\}$

$= [-4/3, 10/3]$

$f^{-1}([-6,5]) = \{x \in R \mid f(x) \in [-6,5]\}$

$= \{x \in R \mid -6 \le f(x) \le 5\}$

- When $x > 0$, we have $f(x) = 3x - 5$

$\therefore -6 \le f(x) \le 5$

$\Rightarrow -6 \le (3x-5) \le 5$

$\Rightarrow -1 \le 3x \le 10$

$\Rightarrow -\frac{1}{3} \le x \le \frac{10}{3}$

- When $x \le 0$, we have $f(x) = -3x + 1$

$\therefore -6 \le f(x) \le 5$

$\Rightarrow -6 \le (-3x+1) \le 5$

$\Rightarrow -7 \le -3x \le 4$

$\Rightarrow \frac{7}{3} \ge x \ge -\frac{4}{3}$ or $-\frac{4}{3} \le x \le \frac{7}{3}$

$\therefore f^{-1}([-6,5]) = \{x \in R \mid -\frac{4}{3} \le x \le \frac{7}{3}$ or $-\frac{1}{3} \le x \le \frac{10}{3}\}$

$= \{x \in R \mid -\frac{4}{3} \le x \le \frac{10}{3}\}$

$f^{-1}([-6,5]) = [-4/3, 10/3]$

(2) If $A = \{1, 2, 3, 4, 5\}$ and there are 6720 injective function $f : A \to B$, What is $|B|$?

(or)

If $|A| = 5$, find $|B|$ given that the number of injective functions from A to B is 6720.

Sol: If $|A| = m$ and $|B| = n$, we know that the number of one-to-one functions from A to B is $P(n, m) = \dfrac{n!}{(n-m)!}$

Given, $|A| = 5$ and $P(n, 5) = 6720$, $n = ?$

$$\because \quad P(n, 5) = \frac{n!}{(n-5)!}$$

$$\Rightarrow 6720 = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)!}{(n-5)!}$$

$$\Rightarrow 6720 = n(n-1)(n-2)(n-3)(n-4)$$

If $n = 6$, $RHS = (6)(5)(4)(3)(2) = 720 \neq LHS$

If $n = 7$, $RHS = (7)(6)(5)(4)(3) = 2520 \neq LHS$

If $n = 8$, $RHS = (8)(7)(6)(5)(4) = 6720 = RHS$.

Hence $n = 8$

Thus the required $|B| = 8$

## (*) The pigeonhole principle

- If 'm' pigeon occupy 'n' pigeonholes and if $m > n$ then at least one pigeonhole contains two or more pigeons in it.

- Generalization

- If 'm' pigeons occupy 'n' pigeon holes, and if $m > n$ then atleast one pigeonhole must contain $(p+1)$ or more pigeons

where, $p = \left\lfloor \frac{m-1}{n} \right\rfloor$

or

$p =$ flooring of $\frac{m-1}{n}$

# Worked problems

1. State pigeon hole principle. Prove that in any set of 29 persons, at least 5 persons have been born on the same day of the week.

**Sol:** • If 'm' pigeons occupy 'n' pigeon holes and if $m > n$, then at least one pigeon hole must contain 2 or more pigeons in it.

• If 'm' pigeons occupy 'n' pigeon holes, then atleast one pigeon hole must contain $(p+1)$ or more pigeons where,

$$p = \left\lfloor \frac{m-1}{n} \right\rfloor$$

• Given, $m = 29$, $n = 7$ (no of days in a week)

$$\therefore p = \left\lfloor \frac{m-1}{n} \right\rfloor = \left\lfloor \frac{29-1}{7} \right\rfloor = \left\lfloor \frac{28}{7} \right\rfloor = 4$$

$$\Rightarrow \boxed{p = 4}$$

$$\therefore p+1 = 4+1 = 5$$

2 prove that if any number from 1 to 8 are chosen then two of them will have their sum as 9.

S:- ~~[illegible]~~ Let, sum of $9 = \{(1,8)(2,7)(3,6)(4,5)\}$ are the 4 sets whose sum is 9 from 1 to 8.

$\therefore m = 8, \; n = 4$

$$p = \left\lfloor \frac{m-1}{n} \right\rfloor = \left\lfloor \frac{8-1}{4} \right\rfloor = \left\lfloor \frac{7}{4} \right\rfloor = \lfloor 1.7 \rfloor = 1$$

$\therefore p+1 = 1+1 = 2$

3 Show that if seven numbers are selected 1 to 12, then two of them will add upto 13.

S:- Let $S = \{1, 2, 3, ----, 12\}$

Two numbers from S adding to 13 are the sets $= \{(1,12), (2,11) (3,10) (4,9) (5,8) (6,7)\}$ are 6 sets

$m = 12, \quad n = 6$

$$p = \left\lfloor \frac{m-1}{n} \right\rfloor = \left\lfloor \frac{12-1}{6} \right\rfloor = \left\lfloor \frac{11}{6} \right\rfloor = \lfloor 1.8 \rfloor = 1$$

$\therefore p+1 = 1+1 = 2$

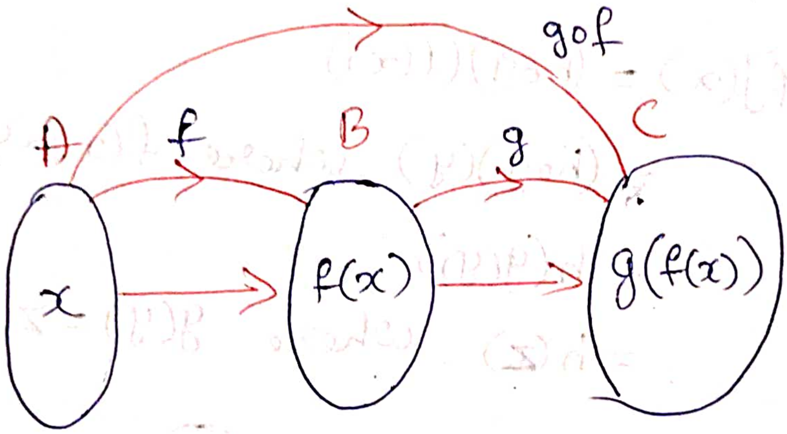
~~(*) Composition of funct~~

## *) Function Composition and Inverse Functions

### → Composition of two functions

Consider 3 non-empty sets A, B, C and the functions $f: A \to B$ and $g: B \to C$. The composition of these 2 functions is defined as the function

$$g \circ f : A \to C \text{ with } (g \circ f)(x) = g(f(x)), \ \forall x \in A$$




**Note:-**

① $f: A \to A$ then $f \circ f = f^2$

$f \circ f^2 = f^3$

② $(g \circ f)(x) = g(f(x))$

③ $(f \circ g)(x) = f(g(x))$

## Worked problems

(1) If $f: A \to B$, $g: B \to C$, $h: C \to D$ are three functions, then prove that $(h\circ g)\circ f = h\circ(g\circ f)$

Proof: Let, $g: B \to C$ and $h: C \to D \Rightarrow h\circ g: B \to D$

$f: A \to B$ and $g: B \to C \Rightarrow g\circ f: A \to C$

$\therefore (h\circ g)\circ f: A \to D$ and $h\circ(g\circ f): A \to D$

Let $x \in A$ and we have,

$$[(h\circ g)\circ f](x) = (h\circ g)(f(x))$$
$$= (h\circ g)(y), \text{ where } f(x) = y,\ y \in B.$$
$$= h(g(y))$$
$$= h(z), \text{ where } g(y) = z,\ z \in C$$

$$[(h\circ g)\circ f](x) = h(z) \quad \text{--- (1)}$$

Now,

$$[h\circ(g\circ f)](x) = h[(g\circ f)(x)]$$
$$= h[g(f(x))]$$
$$= h[g(y)]$$
$$= h(z)$$

$$\therefore [(h\circ g)\circ f](x) = h(z) \quad \text{--- (2)}$$

Comparing ① and ② we have

$$[(hog)of](x) = [ho(gof)](x)$$

$$\Rightarrow (hog)of = ho(gof)$$

② If $A=\{1,2,3,4\}$; $B=\{a,b,c\}$; $C=\{w,x,y,z\}$ with $f: A \to B$ and $g: B \to C$ given by $f=\{(1,a)(2,a)(3,b)(4,c)\}$ and $g=\{(a,x)(b,y)(c,z)\}$, find gof.

S:- Let, $(gof)(x) = g(f(x))$, $\forall x \in A$. we have by data $f(1)=a$, $f(2)=a$, $f(3)=b$, $f(4)=c$ and $g(a)=x$, $g(b)=y$, $g(c)=z$

$$\therefore (gof)(1) = g(f(1)) = g(a) = x$$
$$(gof)(2) = g(f(2)) = g(a) = x$$
$$(gof)(3) = g(f(3)) = g(b) = y$$
$$(gof)(4) = g(f(4)) = g(c) = z$$

Thus, $gof = \{(1,x)(2,x)(3,y)(4,z)\}$

## (*) Identity function

If $A$ is a non empty set and $f: A \to A$ is a function such that every element of $A$ is mapped onto itself, then $f$ is called an Identity function denoted by $I_A$.

i.e., $f(x) = x$, $\forall x \in A$

$$I_A = \{(x, x) / x \in A\} ; I_A(x) = x, \forall x \in A$$

Note:- $f \circ I_A = f = I_B \circ f$

## (*) Invertible functions

- A function $f: A \to B$ is said to be invertible if and only if $f^{-1}: B \to A$ is also a function.

(or)

- A function $f: A \to B$ is said to be invertible if there exists a function $g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$. Here $I_A$ and $I_B$ are respectively the identity function on $A$ and $B$, $g \circ f$ and $f \circ g$ are functions from $A$ to $A$ and $B$ to $B$ respectively. The function $g: B \to A$ is called the inverse of $f$ and is represented by $g = f^{-1}$.

## Worked problems

1. Let $A = B = C = R$ and $f: A \to B$, $g: B \to C$ be defined by $f(a) = 2a+1$, $g(b) = \frac{1}{3}b$, $\forall a \in A, \forall b \in B$. Compute $g \circ f$ and show that $g \circ f$ is invertible. What is $(g \circ f)^{-1}$?

**Sol:-** Given, $f(a) = 2a+1$, $g(b) = \frac{1}{3}b$, $\forall a \in A, \forall b \in B$.

Consider, $(g \circ f)(a) = g(f(a))$

$$= g(2a+1)$$

$$(g \circ f)(a) = \frac{2a+1}{3}$$

Suppose, $(g \circ f)(a) = c$, we have $c = \frac{2a+1}{3}$

i.e., $3c = 2a+1$ or $a = \frac{3c-1}{2}$

But $(g \circ f)(a) = c \Rightarrow a = (g \circ f)^{-1}(c)$ and $a = \frac{3c-1}{2} \in R$

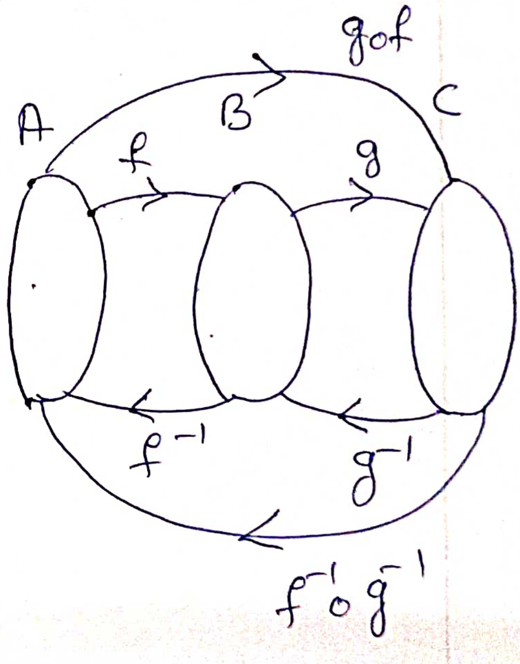
$\therefore$ we conclude that $(g \circ f)^{-1}$ exists

Also, $(g \circ f)^{-1}(c) = \frac{3c-1}{2}$

2 If $f: A \to B$ and $g: B \to C$ are invertible then (i) $gof$ is also invertible

(ii) $(gof)^{-1} = f^{-1} o g^{-1}$

Sol:- (i) Given, $f: A \to B$ and $g: B \to C$ are invertible functions they are both one-one and onto functions. Therefore $gof: A \to C$ is also one-one and onto function. Hence $gof$ is also invertible function.




(ii) Let,

$f: A \to B$ and $g: B \to C$

$\therefore gof: A \to C$ and $(gof)^{-1}: C \to A$

$g^{-1}: C \to B$ and $f^{-1}: B \to A$

$\therefore f^{-1} o g^{-1}: C \to A$

Consider,

$(gof)^{-1}(c) = a$, where $a \in A$ and $c \in C$ — (1)

$\Rightarrow c = (gof)(a) = g(f(a))$

$\Rightarrow c = g(f(a))$ or $g^{-1}(c) = f(a)$ or $f^{-1}(g^{-1}(c)) = a$

i.e., $(f^{-1} o g^{-1})(c) = a$ — (2)

From (1) and (2) we have, $(gof)^{-1}(c) = (f^{-1} o g^{-1})(c) \; \forall \; c \in C$

$\Rightarrow (gof)^{-1} = f^{-1} o g^{-1}$